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Asymptotically lacunary equivalent sequences defined by ideals and modulus function

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Abstract

Purpose: For a non-trivial ideal $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$, a lacunary sequence $\theta = (k_r)$, and a modulus function f, the purpose of present paper is to introduce certain new notions: $\mathcal{I}(f)$-asymptotically equivalent, $\mathcal{I}(w_f)$-asymptotically equivalent, and $\mathcal{I}(N_\theta^f)$-asymptotically equivalent sequences of numbers.

Methods: We use an analytic method to obtain our results.

Results: Certain theorems on generalized equivalent sequences by the use of ideals, lacunary sequences, and a modulus function are obtained.

Conclusions: We observe that if the modulus function f satisfies $\lim_{t\to\infty} \frac{f(t)}{t} > 0$, then the notions $\mathcal{I}(w)$ and $\mathcal{I}(N_\theta)$ respectively coincide with the notions $\mathcal{I}(w_f)$ and $\mathcal{I}(N_\theta^f)$. However, if f is bounded, the notions $\mathcal{I}(w_f)$ and $\mathcal{I}(N_\theta^f)$ coincides respectively with the notions $\mathcal{I}(S)$ and $\mathcal{I}(S_\theta)$.

Keywords: Statistical convergence, $\mathcal{I}$-convergence, Lacunary sequences, Modulus function

AMS subject classification: 40A05; 40A99

Introduction

The idea of statistical convergence for number sequences is introduced by Fast [1] and later developed by [2-6] and many others.

Definition 1.1. [1] A number sequence $x = (x_k)$ is said to be statistically convergent to a number L (denoted by $S - \lim_{k\to\infty} x_k = L$) provided that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| = 0,$$

where the vertical bars denote the cardinality of the enclosed set.

By a lacunary sequence, we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Let, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. Using lacunary sequences, Fridy et al. [7] defined S_θ-convergence, a generalized statistical convergence as follows.

Definition 1.2. [7] Let $\theta = (k_r)$ be a lacunary sequence. A sequence $x = (x_k)$ of numbers is said to be lacunary statistically convergent to a number L (denoted by $S_\theta - \lim_{k\to\infty} x_k = L$) if for each $\epsilon > 0$,

$$\lim_{r\longrightarrow\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0.$$

In order to compare the rate of growth of two sequences, Marouf [8] defined asymptotically equivalent sequences of real numbers and studied its relations with certain matrix-transformed sequences. Patterson et al. [9] studied the asymptotically lacunary statistical equivalent analog of these ideas. Subsequently, many authors have shown their interest to solve different problems arising in this area (see [10-16]). In this work, we define asymptotically equivalent sequences using lacunary sequences, ideals and a modulus function and obtain some revelent connections between these notions.

Definition 1.3. [8] The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent of multiple L provided that

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$$\lim_{k\to\infty}\left(\frac{x_k}{y_k}\right) = L,$$

(denoted by $x \sim y$) and is called simply asymptotically equivalent if $L = 1$.

For any non-empty set X, let $\mathcal{P}(X)$ denote the power set of X.

Definition 1.4. A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is said to be an ideal in X if

(1) $\emptyset \in \mathcal{I}$;
(2) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$ and
(3) $A \in \mathcal{I},\ B \subset A$ imply $B \in \mathcal{I}$.

Definition 1.5. A non-empty family $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be a filter in X if

(1) $\emptyset \notin \mathcal{F}$;
(2) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$ and
(3) $A \in \mathcal{F},\ B \supset A$ imply $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if it contains all the singleton sets.

Moreover, if $\mathcal{I}$ is a non-trivial ideal on X, then $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on X and conversely. The filter $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

Using ideals, Kostyrko et al. [17] defined $\mathcal{I}$-convergence, a stronger convergence in a metric space, whereas Dass et al. [18] unified this idea with statistical convergence for real sequences.

Definition 1.6. [17] Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and (X, ρ) be a metric space. A sequence $x = (x_k)$ in X is said to be $\mathcal{I}$-convergent to ξ if for each $\epsilon > 0$, the set

$$A(\epsilon) = \{k \in \mathbb{N} : \rho(x_k, \xi) \geq \epsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{k\to\infty} x_k = \xi$.

Definition 1.7. [18] A sequence $x = (x_k)$ of numbers is said to be $\mathcal{I}$-statistical convergent or $S(\mathcal{I})$-convergent to L, if for every $\epsilon > 0$ and $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |x_k - L| \geq \epsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

In this case, we write $x_k \rightarrow L(S(\mathcal{I}))$ or $S(\mathcal{I}) - \lim_{k\to\infty} x_k = L$.

Nakano [19] introduced the notion of a modulus function in 1953 as follows. By a modulus function, we mean a function f from $[0, \infty)$ to $[0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$;
(2) $f(x + y) \leq f(x) + f(y)$ for all $x \geq 0, y \geq 0$;
(3) f is increasing;
(4) f is continuous from the right at 0.

It follows that f must be continuous on $[0, 1)$. A modulus may be bounded or unbounded. Many authors, including Connor [20], Kolk [21], Maddox [22], Öztürk et al. [23], Pehlivan et al. [24,25] and many others used a modulus f to construct some sequence spaces.

Recently, Bilgin [26] used modulus function to define some notions of asymptotically equivalent sequences and studied some of their connections. We now consider some new kind of asymptotically equivalent sequences defined by ideals, lacunary sequences and a modulus function.

Methods

We use an analytic method to obtain our results.

Results and discussion

We now consider our main results. We begin with the following definitions.

Definition 2.1. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(w)} y\right)$ and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2.2. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$ and $\gamma > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(S_\theta)} y\right)$ and simply asymptotically lacunary statistical equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2.3. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(N_\theta)} y\right)$ and simply strongly asymptotically lacunary equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2.4. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and f be a modulus function. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$,

$$\left\{ k \in \mathbb{N} : f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \epsilon \right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(f)} y\right)$ and simply f-asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2.5. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$ and f be a modulus function. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \epsilon \right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(w_f)} y\right)$ and simply strongly f-asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Definition 2.6. Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, f be a modulus function and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for each $\epsilon > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \epsilon \right\} \in \mathcal{I},$$

denoted by $\left(x \sim^{\mathcal{I}(N_\theta^f)} y\right)$ and simply strongly f-asymptotically lacunary equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

Lemma 2.1. *[10,25] Let f be a modulus function and let $0 < \delta < 1$. Then for $y \neq 0$ and each $\left(\frac{x}{y}\right) > \delta$, we have $f\left(\frac{x}{y}\right) \leq \frac{2f(1)}{\delta}\left(\frac{x}{y}\right)$.*

Theorem 2.1. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, and f be a modulus function. Then,*

(1) if $x \sim^{\mathcal{I}(w)} y$ then $x \sim^{\mathcal{I}(w_f)} y$ and
(2) $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then $x \sim^{\mathcal{I}(w)} y \Leftrightarrow x \sim^{\mathcal{I}(w_f)} y$.

Proof. (1) Let $x \sim^{\mathcal{I}(w)} y$ and $\epsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. We can write

$$\begin{aligned}\frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) &= \frac{1}{n} \sum_{1} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \\ &\quad + \frac{1}{n} \sum_{2} f\left(\left| \frac{x_k}{y_k} - L \right| \right),\end{aligned}$$

where the first summation runs over $\left| \frac{x_k}{y_k} - L \right| \leq \delta$, and the second summation on $\left| \frac{x_k}{y_k} - L \right| > \delta$. Moreover, using the definition of the modulus function f, we have

$$\frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) < \epsilon + \left(\frac{2f(1)}{\delta} \right) \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right|.$$

Thus, for any $\eta > 0$,

$$\begin{aligned}&\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \eta \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \frac{(\eta - \epsilon)\delta}{2f(1)} \right\}.\end{aligned}$$

Since $x \sim^{\mathcal{I}(w)} y$, it follows the later set, and hence, the first set in above expression belongs to $\mathcal{I}$. This proves that $x \sim^{\mathcal{I}(w_f)} y$. □

(2) If $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then we have $f(t) \geq \alpha t$ for all $t > 0$. Suppose that $x \sim^{\mathcal{I}(w_f)} y$. Since

$$\begin{aligned}\frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) &\geq \frac{1}{n} \sum_{k=1}^{n} \alpha \left(\left| \frac{x_k}{y_k} - L \right| \right) \\ &= \alpha \left(\frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \right),\end{aligned}$$

it follows that for each $\epsilon > 0$, we have

$$\begin{aligned}&\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \\ &\subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \alpha\, \epsilon \right\}.\end{aligned}$$

Since $x \sim^{\mathcal{I}(w_f)} y$, it follows that the later set belongs to $\mathcal{I}$, and therefore, the theorem is proved.

Theorem 2.2. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, and f be a modulus function. Then,*

(1) if $x \sim^{\mathcal{I}(w_f)} y$, then $x \sim^{\mathcal{I}(S)} y$ and
(2) if f is bounded, then $x \sim^{\mathcal{I}(w_f)} y \Leftrightarrow x \sim^{\mathcal{I}(S)} y$.

Proof. (1) Suppose $x \sim^{\mathcal{I}(w_f)} y$, and let $\epsilon > 0$ be given, then we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \frac{1}{n}\sum_{\substack{k=1;\\ \left|\frac{x_k}{y_k}-L\right|\geq\epsilon}}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$
$$\geq \frac{f(\epsilon)}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k}-L\right| \geq \epsilon\right\}\right|.$$

□

Consequently, for any $\eta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k}-L\right| \geq \epsilon\right\}\right| \geq \frac{\eta}{f(\epsilon)}\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \eta\right\}.$$

Since $x \sim^{\mathcal{I}(w_f)} y$, it follows by Definition 2.5 that the later set belongs to $\mathcal{I}$, and therefore, $x \sim^{\mathcal{I}(S)} y$.

(2) Suppose f is bounded and $x \sim^{\mathcal{I}(S)} y$. Since f is bounded, there exists a real number M such that $\sup f(t) \leq M$. Moreover, for $\epsilon > 0$, we can write

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right) = \frac{1}{n}\left[\sum_{\substack{k=1\\ \left|\frac{x_k}{y_k}-L\right|\geq\epsilon}}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right) + \sum_{\substack{k=1\\ \left|\frac{x_k}{y_k}-L\right|<\epsilon}}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right)\right]$$
$$\leq \frac{M}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k}-L\right| \geq \epsilon\right\}\right| + f(\epsilon).$$

Now on applying the operators $\epsilon \to 0$, the result follows similarly as in case of (1).

Theorem 2.3. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $\theta = (k_r)$ be a lacunary sequence and f be a modulus function. If $\liminf_r q_r > 1$, then $x \sim^{\mathcal{I}(w_f)} y \Rightarrow x \sim^{\mathcal{I}(N_\theta^f)} y$.*

Proof. Suppose $\liminf_r q_r > 1$, then there exist $\delta > 0$ such that $q_r = \frac{k_r}{k_{r-1}} \geq 1+\delta$. This implies that $\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}$. Let $x \sim^{\mathcal{I}(w_f)} y$. For a sufficiently large r, we obtain the following:

$$\frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \frac{1}{k_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$
$$= \left(\frac{h_r}{k_r}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$
$$\geq \left(\frac{\delta}{1+\delta}\right)\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right),$$

which gives for any $\epsilon > 0$,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \epsilon\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \frac{1}{k_r}\sum_{k=1}^{k_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \frac{\epsilon.\delta}{1+\delta}\right\}.$$

□

Since $x \sim^{\mathcal{I}(w_f)} y$, it follows that the later set and, hence, the former set belongs to $\mathcal{I}$. This shows that $x \sim^{\mathcal{I}(N_\theta^f)} y$.

Theorem 2.4. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $\theta = (k_r)$ be a lacunary sequence and f be a modulus function. Then,*

(1) *if $x \sim^{\mathcal{I}(N_\theta)} y$, then $x \sim^{\mathcal{I}(N_\theta^f)} y$; and*
(2) *$\lim_{t\to\infty}\frac{f(t)}{t} = \alpha > 0$, then $x \sim^{\mathcal{I}(N_\theta)} y \Leftrightarrow$ $(x \sim^{\mathcal{I}(N_\theta^f)} y$.*

Proof. The proof is similar to the proof of Theorem 2.1, so we omit it here. □

Theorem 2.5. *Let $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ be a non-trivial ideal in $\mathbb{N}$, $\theta = (k_r)$ be a lacunary sequence and f be a modulus function. Then,*

(1) if $x \sim^{\mathcal{I}(N_\theta^f)} y$, then $x \sim^{\mathcal{I}(S_\theta)} y$;
(2) if f is bounded, then $x \sim^{\mathcal{I}(N_\theta^f)} y \Leftrightarrow x \sim^{\mathcal{I}(S_\theta)} y$.

Proof. (1) Suppose $x \sim^{\mathcal{I}(N_\theta^f)} y$, and let $\epsilon > 0$ be given. Since

$$\frac{1}{h_r}\sum_{k\in I_r} f\left(\left|\frac{x_k}{y_k}-L\right|\right) \geq \frac{1}{h_r}\sum_{\substack{k\in I_r\\ \left|\frac{x_k}{y_k}-L\right|\geq\epsilon}} f\left(\left|\frac{x_k}{y_k}-L\right|\right)$$
$$\geq f(\epsilon)\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k}-L\right| \geq \epsilon\right\}\right|,$$

it follows that for any $\gamma > 0$, if we denote sets

$$A(\epsilon,\gamma) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| \geq \gamma \right\}$$

$$B(\epsilon,\gamma) \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \geq \gamma f(\epsilon) \right\},$$

then $A(\epsilon,\gamma) \subseteq B(\epsilon,\gamma)$. Since $x \sim^{\mathcal{I}(N_\theta^f)} y$, so $B(\epsilon,\gamma) \in \mathcal{I}$. But then, by definition of an ideal, $A(\epsilon,\gamma) \in \mathcal{I}$, and therefore, $x \sim^{\mathcal{I}(S_\theta)} y$. □

(2) Suppose that f is bounded, and let $x \sim^{\mathcal{I}(S_\theta)} y$. Since f is bounded, there exists a positive real number M such that $|f(x)| \leq M$ for all $x \geq 0$. Further, using the fact

$$\frac{1}{h_r} \sum_{k \in I_r} f\left(\left| \frac{x_k}{y_k} - L \right| \right) = \frac{1}{h_r} \left[\sum_{\substack{k \in I_r \\ \left| \frac{x_k}{y_k} - L \right| \geq \epsilon}} f\left(\left| \frac{x_k}{y_k} - L \right| \right) + \sum_{\substack{k \in I_r \\ \left| \frac{x_k}{y_k} - L \right| < \epsilon}} f\left(\left| \frac{x_k}{y_k} - L \right| \right) \right]$$

$$\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| + f(\epsilon),$$

the proof can be obtained on the same lines as that for part (2) of Theorem 2.2.

Conclusions

We observe that if the modulus function f satisfies $\lim_{t \to \infty} \frac{f(t)}{t} > 0$, then the notions $\mathcal{I}(w)$ and $\mathcal{I}(N_\theta)$ respectively coincide with the notions $\mathcal{I}(w_f)$ and $\mathcal{I}(N_\theta^f)$. However, if f is bounded, the notions $\mathcal{I}(w_f)$ and $\mathcal{I}(N_\theta^f)$ coincides respectively with the notions $\mathcal{I}(S)$ and $\mathcal{I}(S_\theta)$.

Competing interests
Both authors declare that they have no competing interests.

Authors' contributions
Both authors contributed to the article equally. Both authors also read and approved the final manuscript for publication.

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